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Einstein–A Natural Completion of Newton

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It is generally accepted that the theory of relativity is the greatest contribution to scientific thought that has emanated from a single mind since Newton's formulation of the laws of motion. At the same time Einstein's contributions have been interpreted as a departure from Newtonian ideas and this belief is essentially due to the fact that new concepts like the equivalence of mass and energy and the symmetry of spacetime were not envisioned in the Newtonian universe. We shall now present the relativistic theory as a natural continuance and completion of Newtonian ideas. The mansion of relativity has many entrances and the most suitable one for entering it from the Newtonian structure is the *velocity transformation formula.*

We start with the following simple argument which takes us right into the heart of the theory of special relativity. Considering one dimensional motion if v is real and is a possible velocity of a point particle so is $-v$ since it merely implies a reversal in direction. Accepting the Newtonian definition of relative velocity and the axiomatic principle of no preference for any particular frame we find $2v$ is a realizable velocity. Therefore $2^n v$ is also realizable, where n can be chosen as large as we please. If we make the postulate that such a world admitting velocities as large as we please would be "chaotic," then an upper limit l has to be prescribed for the relative velocity. Thus if v_a and v_b are velocities of two point particles a and b, then the relative velocity is assumed to be

$$v_r = (v_a - v_b)/f(v_a, v_b). \tag{1}$$

We now require that

$$v_r < l \quad \text{if} \quad v_a < l \quad \text{and} \quad v_b < l \tag{2}$$

and that

$$\begin{aligned} v_r &\to v_a - v_b \quad \text{as} \quad v_a \quad \text{and} \quad v_b \to 0 \\ v_r &\to l \quad \text{as} \quad v_a \quad \text{and (or)} \quad v_b \to l. \end{aligned} \tag{3}$$

The only choice of $f(v_a, v_b)$ turns out to be

$$1 - v_a v_b / l^2. \tag{4}$$

We find that on setting $l = 1$ without loss of generality the relative velocity formula is given by

$$v_r = (v_a - v_b)/(1 - v_a v_b). \tag{5}$$

Equivalently

$$v_a = (v_r + v_b)/(1 + v_r v_b); \qquad v_b = (v_a - v_r)/(1 - v_a v_r). \tag{6}$$

v_a and v_b can be considered velocities of particles a and b relative to a particle c. Thus (5) and (6) demonstrate the symmetry among the velocities v_a, v_b, v_r on the axiomatic principle that all velocities are relative.

Let us define the following pairs of quantities

$$\xi_r = v_r/\sqrt{1 - v_r^2}; \qquad \eta_r = 1/\sqrt{1 - v_r^2}. \tag{7}$$

Substituting for ξ_a and η_a in (5) and using the simple identity

$$(1 - v_a v_b)^2 - (v_a - v_b)^2 \equiv (1 - v_a^2)(1 - v_b^2), \tag{8}$$

we find

$$\begin{pmatrix} \xi_r \\ \eta_r \end{pmatrix} = \begin{pmatrix} \eta_b & -\xi_b \\ -\xi_b & \eta_b \end{pmatrix} \begin{pmatrix} \xi_a \\ \eta_a \end{pmatrix}. \tag{9}$$

We immediately recognise this to be the Lorentz transformation for space and time or momentum and energy, if we define

$$\begin{aligned} x_r \quad \text{or} \quad p_r &= K\xi_r; \qquad t_r \quad \text{or} \quad E_r = K\eta_r; \\ x_a \quad \text{or} \quad p_a &= K\xi_a; \qquad t_a \quad \text{or} \quad E_a = K\eta_a, \end{aligned} \tag{10}$$

where K is a constant independent of the velocities. To establish the correspondence we have assumed that x and t are such that $|x/t| < 1$, that is we are considering events separated in space and time coordinates in such a manner that they can be connected by a signal with velocity less than unity or equivalently with the motion of a *single particle* with uniform velocity $v = x/t = p/E$. Thus, writing

$$t^2 - x^2 = K^2, \tag{11}$$

K is identified to be the "proper time". If we write

$$E^2 - p^2 = K^2, \tag{12}$$

K is the mass of the point particle in a Lorentz transformation.

The transformation laws have hitherto been obtained on the assumption that the three ratio's ξ_r/η_r, ξ_a/η_a and ξ_b/η_b are less than one. However these relations can be analytically continued to the case when any *two* of these three ratios are greater than unity while the third is less than unity. In such a case the ***two ratios which are greater than unity cannot be interpreted as velocities*** while the third one retains such an interpretation. We now observe that for example if we assume that

$$\begin{gathered} | x_a/t_a | = | \xi_a/\eta_a | > 1, \\ | x_r/t_r | = | \xi_r/\eta_r | > 1, \qquad \text{and} \qquad | \xi_b/\eta_b | < 1 \end{gathered} \tag{13}$$

we can still treat Eq. (9) as valid without giving a physical interpretation of the ratios ξ_a/η_a and ξ_r/η_r as velocities. However the equation is not valid if all the three ratios are greater than unity or only one of them is greater and the other two less than unity.

Such a procedure is not unfamiliar to a mathematician and in fact has been the established mode of creating new mathematical structures from existing ones. For example, in number theory, starting with positive integers we introduce the operation of addition. Defining subtraction as the inverse of addition we find that it necessiates the inclusion of negative integers. Similarly the definition of division as the inverse of multiplication leads to fractional numbers. Irrational numbers are obtained as the limits of rational sequences and thus we are able to "fill" the entire real line.

In much the same spirit we now interpret the variables x and t though $x/t = \xi/\eta > 1$. Since the velocity concept breaks down if $\xi/\eta > 1$, it is quite clear that x and t have to be interpreted as the spatial and temporal separation of events which are not the result of one another or have no causal connection between them. In other words, *they refer to two particles which exist independently of one another*. Since a time difference may exist in some frame of reference we cannot speak of the simultaneity of the events but only of independence or acausality. Such an extension of the Lorentz transformation to values of x and t such that $x/t > 1$ thus implies the independent existence of two particles, a concept which has to be *postulated* in a Newtonian theory but which can be *derived* as an analytic continuation in the Einstein approach. If matter itself is interpreted as the collection of independently existing point particles distributed in space, the existence of matter in bulk seems to be a natural consequence of the existence of space like intervals. To identify two points in space we need the independent existence of two particles or in other words *we need matter to identify space*. Though matter may not *fill* the whole of space, it can *span* the whole of it when viewed from moving frames of reference.

A similar interpretation can be carried over to a description in the language of momentum and energy. The existence of two independent particles should not be interpreted as events relating to a single particle moving faster than light. This amounts to the assumption of the spontaneous splitting of one particle into two which is thought to be a test of tachyons. Far from being so it leads to the conceptual difficulties of considering two independently existing particles as one and the same.

Such difficulties arise due to the over-emphasis on the symmetry of space and time coordinates. However the concept of velocity itself is not symmetrical in space and time, since time is preferred as the denominator and the spatial coordinates as numerators.

In the case $x/t > 1$ the vanishing of time interval in a suitably chosen frame would imply that the ratio x/t becomes infinite and therefore cannot be interpreted as a velocity since we have started with the postulate that infinite velocities are not realisable. Such a difficulty does not arise in the case when the numerator or the spatial interval vanishes in suitably chosen frames of reference when considering events for which $x/t < 1$.

Moreover if we wish to study multidimensional spaces, the increase in the number of dimensions is possible only in space coordinates while there can be *one and only one time coordinate*. This conclusion emerges as a natural consequence of the L-matrix theory systematically developed by the author in recent years.

It is a well known panegyric on Newton:

The Lord said, "Let Newton be," and all was light.

For Einstein's theory, it may read:

The Lord said, "Let there be light"—all else is space, time and matter.